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A stochastic approach to the Euler–Poincaré number of the loop space of a developable orbifold

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Abstract

We define a regularised version of the de Rham operator over the free loop space. We perform a semi-classical approximation of it, such that the index of the limit operator is equal to the "orbit Euler characteristic" of physicists.

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0. Introduction

In the physics of string theory, one considers string propagation on a manifold M quotiented by a finite group of symmetries G. When the group action is not free, the quotient space M/G is in general not a smooth manifold, but one with singularities, a so-called developable orbifold. In the discussion of string vacua for M/G, one has to consider the configuration of the closed (parametrised) loops of M together with all the loops twisted by elements of G. By consideration of modular invariance of the theory, Dixon, Harvey, Vafa and Witten [DHVW] introduced the following "orbifold Euler characteristic" of the quotient of M by G as the appropriate Euler number for the purpose of string theory:

$$\chi(M, G) = \frac{1}{|G|} \sum \chi(M^{\langle g,h \rangle}) , \tag{0.1}$$

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where the summation runs over all commuting pairs in $G\times G$, and $M^{\langle g,h\rangle}$ denotes the common fixed point set of g and h. For a free action it is a well known fact that $\chi(M, G)=\chi(M/G)$. The connection of this expression with the representation theory of the group G leads to the identification of $\chi(M, G)$ with the Euler characteristic of equivariant K-theory $K_G(M)$, which was noted in [AS]. However, the string calculation of the Euler number is expected to agree with the Euler number of a "correct" resolution M/G^0 of the singular space M/G, at least for manifolds with SU(n)-holonomy. For $\dim_{\mathbb{C}} M=2$, [HH] showed that the equality

$$\chi(M, G)=\chi(M/G^0)\ , \tag{0.2}$$

holds for M/G^0 the minimal resolution of M/G. When $\dim_{\mathbb{C}} M=3$ and G abelian, $\chi(M, G)$ is also identified with $\chi(M/G^0)$ for M/G^0 being the "minimal" toroidal resolution of M/G constructed by the methods in toric geometry in [RY, R], and also in [MOP]. It seems that this phenomenon should hold for a general reasonable class. Even though the formula of the orbifold Euler characteristic was obtained by stringists using physicists' ideas, which is quite natural, it is in some sense unsatisfactory because a clearer mathematical nature of "strings" still relies on a rigorous mathematical description of the intuition behind it. Here we propose a mathematical treatment using a probabilistic method based on Malliavin Calculus, which can justify some intuitive and heuristic methods of the physical arguments. The formulation might shed some light on the "string" nature of toric geometry, which has been a useful device in the study of string compactification.

In order to give an interpretation to these ideas, we need to consider an element of volume over the twisted loop space, and unfortunately we meet the problem that there is no Riemannian measure over the loop space of an orbifold. The idea is to use the twisted B-H-K measure, which extends in the case of twisted loop space the measure which was introduced in [Bi5] in order to explain the relation between the cohomology of the loop space and the index theory (see [HK] in the flat case). In the case of non-twisted loop this measure is used in [JL1] in order to do a L^p theory of Chen forms. But no differential operation is given in [JL1].

Such differential operations have been known for a long time in Malliavin Calculus for the Wiener measure: these are the Malliavin derivatives [Gr1] and the Ornstein–Uhlenbeck operator [Me]. [Sh] and [ArM] study differential forms over the Wiener space and the exterior derivative. [Ar1, Ar2] do an extensive study of the index of the Dirac operator over the flat loop space in the case of the free field or with interacting terms: they give a path integral representation of the index of such operators over the loop, strongly inspired by the work of Hoegh-Krohn in the scalar case [HK] (see [JLW1, JLW2] for physics references).

For the analysis in infinite dimensional curved spaces, there exist different types of theory (we refer to [Ma] for a survey).

– The theory of Wiener manifolds. The reader can see Ramer's thesis [Ra] for the non-scalar case.

– The analysis over infinite dimensional manifolds, which was used by the Russian school. The manifold's structure is very important in such cases [BS, DF].
– The quasi-sure analysis [Ge, Am1], which works over finite codimensional manifolds of the Wiener space (see [K] for forms).
– The analysis over loop groups [AM2, Gr2, Gr3], which is closer to the purpose of this paper, but with a different tangent space, which uses deeply the structure of the group. For the moment there is no manifold structure in this theory nor in the next theory.

The present paper is more related to [Le4] and [Le5], where the case of the free loop space of the Riemannian manifold is considered. Some connections are introduced over the free loop space, integration by parts is done, which allows us to define Malliavin's derivatives of every order and to define an Ornstein–Uhlenbeck operator invariant by rotation.

[JL2] defined a non-equivariant regularised exterior derivative over the full space of forms of the loop space. Its adjoint is computed. A rigorous conjecture for the index of the regularised de Rham operator is given. By localisation, it is the Euler–Poincaré number of the manifold. The situation becomes more complicated for the case of the equivariant Dirac operator over the loop space with the relation with the Witten genus and for the case of the equivariant signature operator over the loop space with the relation with the elliptic genus: some topological obstructions are met [Be, Se, Wi] and in fact in [JL2] there is an extension of the Taubes construction of the Dirac operator over an infinitesimally small loop [T] only over a small neighbourhood, by using stochastic calculus. In order to define the "restriction" to these non-scalar operators to an infinitesimally small loop (that means over the family of Brownian bridges over the set of tangent spaces), the B-H-K measure in small time is introduced and some limit theorems are used, which correspond to the high temperature limit in the stochastic context and which belong to the domain of the computations done in [Bi3, IW, Hs, Le2].

The purpose of this paper is to do analogous computations for the regularised exterior derivative for twisted loop spaces: the loop space of a developable orbifold appears namely as an orbifold of twisted loops. A scalar calculus over each sector of twisted loop is done. A diffusion process is constructed, and some rough localisation is performed for the diffusion process (see [ALR] for non-twisted loops). The big difference with [JL2] is that the limit model is related to the computation of [Bi4] instead of the model of [Bi3], because the twisted loop concentrates on the loop of the fixed point of an element of G in small time.

1. Scalar calculus over twisted loop spaces

1.1. Integration by parts for distinguished vector fields

Let M be a compact Riemannian manifold and let G be a finite group acting over M. By averaging, we can suppose that G is a group of isometries. Let L be the Laplace–Beltrami operator over M and $p_t(x, y)$ the associated heat kernel. $P_{1,x,y}$ is the law of the Brownian bridge starting from x and going to y in time 1. Let H_g be the space of twisted loops going from any x and arriving in gx at time 1. Let μ_g be the measure over H_g,

$$\mathrm{d}\mu_g = p_1(x, gx)\, \frac{P_{1,x,gx}\,\mathrm{d}x}{\int_M p_1(x, gx)\,\mathrm{d}x}\,. \tag{1.1}$$

We denote the associated space of L^2 functions by H_g. Let X_g be the vector field:

$$\tau_t\Big(X_0(\gamma(0)) + \int_{[0,t]} h(s)\,\mathrm{d}s - tX_0(\gamma(0)) + t\tau_1^{-1}\,\mathrm{d}g\,X_0(\gamma(0))\Big) = \tau_t H(t)\,. \tag{1.2}$$

$h(s)$ is equal to $\sum h_i(s)X_i(\gamma(0))$, where each h_i is deterministic such that $\int_{[0,1]} h_i(s)\,\mathrm{d}s = 0$. These vector fields play the role of the distinguished vector fields given in [Le4] and in [Le5]. But the boundary conditions are now $X(1) = \mathrm{d}g\,X(0)$ because we look at twisted loops. Let F be a smooth cylindrical functional $F(\gamma(t(1)), \ldots, \gamma(t(r)))$. We have:

Theorem 1.1.

$$\mu_g[\langle \mathrm{d}F, X_g\rangle] = \mu_g[F \operatorname{div} X_g]\,, \tag{1.3}$$

where

$$\langle \mathrm{d}F, X_g\rangle = \sum \langle \mathrm{d}_{\gamma(t(i))}F(\gamma(t(1)), \ldots, \gamma(t(r))), X_g(t(i))\rangle \tag{1.4}$$

$$\operatorname{div} X_g = \operatorname{div} X_{g,0}(\gamma(0)) + \int_{[0,1]} \langle \tau_s H'(s), \delta\gamma(s)\rangle + \frac{1}{2}\int_{[0,1]} \langle S_{X(s)}, \delta\gamma(s)\rangle - \int_{[0,1]} \operatorname{Tr} s\,\tau_s^{-1} R(\mathrm{d}\gamma(s), \tau_s)\tau_s\tau_1^{-1}\,\mathrm{d}g\,X_{g,0}(\gamma(0))\,, \tag{1.5}$$

where S is the Ricci tensor over M and R the curvature tensor.

Remark. Let us remark that the last term in the divergence can be computed by means of the Ricci tensor and is equal to zero when the manifold is Ricci flat.

Proof. The proof is very similar to the proof of [Le4]. We begin to work over the path space, that means the space of applications from $[0, 1]$ into M endowed with the path space measure $\mathrm{d}x\,P_1^x$, where P_1^x denotes the law of the open Brownian motion at time 1 starting from x. Let ϕ be a smooth function over $M \times M$ with a small support over the diagonal equal to 1 over a small neighbourhood of the diagonal such that $g\gamma(0)$ and $\gamma(1)$ are joined by a unique geodesic if $\phi(g\gamma(0),$

$\gamma(1))$ is not equal to 0. Let us denote by $\tau(\gamma(1), g\gamma(0))$ the parallel transport from $g(\gamma(0))$ to $\gamma(1)$ along this geodesic. We begin by enlarging the vector field X_g over the twisted loop space into a vector field $X_{l,g}$ over the path space by putting:

$$X_{l,g}(t) = \phi(g\gamma(0), \gamma(1))\, \tau_t\Big(X_0(\gamma(0)) + \int_{[0,t]} h(s)\, \mathrm{d}s - tX_0(\gamma(0)) + t\tau_1^{-1}\tau(\gamma(1), g\gamma(0))\, \mathrm{d}g\, X_0(\gamma(0))\Big). \tag{1.6}$$

Let N be a subdivision of $[0, 1]$, and let $X_{l,g}^N$ be the associated vector field and let us consider the polygonal approximation of γ: this polygonal approximation of γ works only if $\gamma(t(i))$ and $\gamma(t(i+1))$ are close, but the contribution of the path where $\gamma(t(i))$ and $\gamma(t(i+1))$ are far goes to 0 when N goes to infinity, as is explained in [Le4, Le5]. We know by integrating by parts in finite dimensions that

$$\nu[\langle \mathrm{d}F, X_{l,g}^N \rangle] = \nu[F \operatorname{div} X_{l,g}^N]\,. \tag{1.7}$$

Moreover, by using the Malliavin Calculus, we know that for all x, y

$$E_{1,x,y}[\langle \mathrm{d}F, X_{l,g}^N \rangle] \to E_{1,x,y}[\langle \mathrm{d}F, X_{l,g} \rangle]\,, \tag{1.8}$$

and that

$$E_{1,x,y}[F \operatorname{div} X_{l,g}^N] \to E_{1,x,y}[F \operatorname{div} X_{l,g}]\,, \tag{1.9}$$

where the divergence is computed from (1.5) by taking the derivative of $\phi(\tau(\gamma(1), g\gamma(0))$ in addition because we are integrating by parts over the path space. By using the fact that g is an isometry, and the relation $X_{l,g}^N(1) = \mathrm{d}g\, X_{l,g}^N(0)$, we arrive at the same cancellation in the end when N goes to infinity as the cancellations registered for non-twisted loops. The only difference is that we do not need to take the derivative of $\tau(\gamma(1), g\gamma(0))$ in the approximation limit procedure, hence the theorem, by considering the matrix from $T_{\gamma(0)}$ into $T_{\gamma(0)}$ by $\tau_1^{-1}\,\mathrm{d}g$ instead of τ_1^{-1} in the case of non-twisted loops. But $\mathrm{d}g\,(\gamma(0))$ has a derivative equal to 0 over a vector field $\tau_t tX$; this explains the fact that no derivative of $\mathrm{d}g$ appears in the last counterterm. □

1.2. Dirichlet form and Ornstein–Uhlenbeck operator

The tangent space is the space of vectors $X(s) = \tau_s H(s)$ with $H(s)$ with finite variations such that $X_1 = \mathrm{d}g\, X_0$. As Hilbert structure, it should be possible to choose the Hilbert structure

$$\|X(0)\|^2 + \int_{[0,1]} \langle DX(s), DX(s) \rangle\, \mathrm{d}s\,,$$

where $DX(s) = \tau_s H'(s)$ is the covariant derivative over the loop. But we will split our tangent space T_γ into $\sum_{\mathbf{Z}} T_\gamma^n$, an orthogonal sum with a different metric in order to simplify the computations.

If $n>0$,

$$T_\gamma^n = \left\{ \tau_t \cdot 2^{1/2} \int_{[0,t]} \cos(ns)\, \mathrm{d}s\, e = X(n,e)(t) \right\} .$$

If $n<0$,

$$T_\gamma^n = \left\{ \tau_t \cdot 2^{1/2} \int_{[0,t]} \sin(ns)\, \mathrm{d}s\, e = X(n,e)(t) \right\} .$$

If $n=0$,

$$T_\gamma^0 = \{ \tau_t (e - te + t\tau_1^{-1}\, \mathrm{d}g\, e) = X(0,e)(t) \} .$$

The Hilbert structure over each piece T_γ^n of T_γ is given by $\|e\|^2_{\gamma(0)}$.

There is a connection which preserves the metric. This arises from the Levi-Civita connection Γ over the manifold:

$$X(n, \Gamma e)(t) = \Gamma(X(n,e)(t)) . \tag{1.10}$$

This connection preserves by definition the splitting of T_γ into T_γ^n.

Let us introduce positive numbers $A(n)$ such $A(n) < C|n|^p \cdot 2p < 1$. Let E' be the following Dirichlet form:

$$E'(F,F) = \sum_{n,i,g} \mu_g [A^2(n) \langle \mathrm{d}F, X(n, e(i)) \rangle^2] , \tag{1.11}$$

where $X(n, e(i))$ is a basis of T_γ^n.

Lemma 1.2. *E' is closed defined over a dense set which separates the twisted loop and tight.*

Proof. E' is densely defined. We choose as core A^0 the set of cylindrical functions $F(\gamma(t(1)), \ldots, \gamma(t(r)))$. Over T_γ^n, $n \neq 0$, we choose as orthonormal basis the natural orthonormal basis which comes from $T_{\gamma(0)}$. We have:

$$|\langle \mathrm{d}F, X(n, e(i)) \rangle| \leqslant C/(|n|+1) . \tag{1.12}$$

Therefore:

$$\sum A(n)^2 \langle \mathrm{d}F, X(n, e(i)) \rangle^2 \leqslant \sum A(n)^2/(|n|+1)^2 \leqslant C , \tag{1.13}$$

with C deterministic. Therefore, E' is densely defined over a set of functions which separate the loop.

– E' is closed. Let us suppose that $F_p \to 0$ for F_p belonging to the core and that:

$$\mu_g\left[\sum A(n)^2(\langle \mathrm{d}F_p, X(n,e(i))\rangle - G(n))^2\right] \to 0\,, \tag{1.14}$$

when $p \to \infty$. Then $G_n = 0$. Namely for all cylindrical functionals F,

$$\begin{aligned}&\mu_g[\langle \mathrm{d}F_p, X(n,e(i))\rangle F]\\ &\quad = \mu_g[F_p \operatorname{div} X_n F] - \mu_g[F_p\langle \mathrm{d}F, X(n,e(i))\rangle]\,,\end{aligned} \tag{1.15}$$

which tends to 0. Therefore $\langle \mathrm{d}F_p, X(n,e(i))\rangle$ tends to 0 in $\mathrm{L}^2(\mu_g)$ and therefore $G_n = 0$ (we used local sections of the smooth orthonormal basis of $T_{\gamma(0)}M$).
– E' is tight for uniform convergence. Let $F(x,y)$ be a smooth function $\geqslant 0$ over $M \times M$ such that $F(x,y) = d^2(x,y)$ if x and y are closed. Let $G(\gamma)$ be the random variable

$$G(\gamma) = \int_{[0,1]}\int_{[0,1]} F(\gamma(s),\gamma(t))^\rho/|t-s|^\alpha\,\mathrm{d}s\,\mathrm{d}t\,, \tag{1.16}$$

which is finite if $\rho > \alpha$. Let us compute $\langle \mathrm{d}G(\gamma), X(n,e(i))\rangle$. It is enough to take $n \neq 0$:

$$\begin{aligned}&|\langle \mathrm{d}G(\gamma), X(n,e(i))\rangle|\\ &\quad < \int_{[0,1]}\int_{[0,1]} F(\gamma(s),\gamma(t))^{\rho-1}/|t-s|^\alpha |\langle \mathrm{d}_{\gamma(s)}F(\gamma(s),\gamma(t)), X(n,e(i))(s)\rangle\\ &\qquad + \langle \mathrm{d}_{\gamma(t)}F(\gamma(s),\gamma(t)), X(n,e(i))(t)\rangle|\,\mathrm{d}s\,\mathrm{d}t\\ &\quad < \frac{C}{|n|}\int_{[0,1]}\int_{[0,1]} F(\gamma(s),\gamma(t))^{\rho-1}/|t-s|^\alpha\,\mathrm{d}s\,\mathrm{d}t\,.\end{aligned} \tag{1.17}$$

Therefore, if $\rho - 1 > \alpha$, $\int_{[0,1]}\int_{[0,1]} F(\gamma(s),\gamma(t))^{\rho-1}/|t-s|^\alpha\,\mathrm{d}s\,\mathrm{d}t$ is finite. Moreover, if we put $\alpha = 1 + 2\beta\rho$, $G < C$ is compact if $\beta < 1/2$ for the uniform norm (see [AV] for the case of a Wiener submanifold). □

The following theorem can be deduced classically from the previous lemma.

Theorem 1.3. *To the Dirichlet form is associated, outside a set of capacity* 0, *a process* $w_t(\gamma)$ *for* μ_g.

Let us consider the operator L_A associated to the Dirichlet form. It has the definition:

$$\begin{aligned}L_A F = &-\sum A(n)^2\langle \mathrm{d}\langle \mathrm{d}F, X(n,e(i))\rangle, X(n,e(i))\rangle\\ &+ \sum A(n)^2\langle \mathrm{d}F, X(n,e(i))\rangle \operatorname{div} X(n,e(i))\,.\end{aligned} \tag{1.18}$$

Theorem 1.4. L_A *is defined over the core* A^0 *if* $4\rho < 1$.

Proof. Only the case $n \neq 0$ is important.

$$|\langle \mathrm{d}\langle \mathrm{d}F, X(n, e(i))\rangle, X(n, e(i))\rangle| \langle C(n)/(n^2+1)\,, \tag{1.19}$$

and the sequence of random variables $C(n)$ is bounded in L^2, this from the relation:

$$\Gamma_X \tau_t = \tau_t \int_{[0,t]} \tau_s^{-1} R(\mathrm{d}\gamma(s), X(s))\tau_s\,, \tag{1.20}$$

for the Levi-Civita connection Γ. So only the second part in the definition of $L_A F$ poses a problem. Let us consider only the $n > 0$ part:

$$\operatorname{div} X(n, e(i)) = \int_{[0,1]} \langle \tau_s \cos(ns)\, e(i), \delta\gamma(s)\rangle$$
$$+\tfrac{1}{2} \int_{[0,1]} \langle S_{X(n,e(i))(s)}, \delta\gamma(s)\rangle + \text{counterterms}\,. \tag{1.21}$$

The counterterms have a behaviour in $C(n)/n$ with $C(n)$ uniformly bounded in $\mathrm{L}^2(\mu_g)$ and do not pose any problem.

Let us consider the jth part of the derivative of $F(\gamma(t(1)), ..., \gamma(t(r)))$. Let us consider the element of $\mathrm{L}^2[0, 1]$ whose Fourier series is 0 if $n < 0$ and $(A(n)^2/n)(\sin(nt(j)) - 1)$. Denote it by $h_{t(j)}(s)$. (The convergence is obtained because $4\rho < 1$.) In the first contribution of the divergence in the operator, we recognise:

$$\int_{[0,1]} \langle \tau_s \langle \mathrm{d}_{\gamma(t(j))} F(\gamma(t(1)), ..., \gamma(t(r))), \tau_{t(j)} h_{t(j)}(s) e(i)\rangle, \delta\gamma(s)\rangle\,, \tag{1.22}$$

which belongs in $\mathrm{L}^2(\mu_g)$ because we recognise a non-anticipative Itô integral. □

Lemma 1.5. *Let F be cylindrical functional.*

$$\mu_g[\exp[C|L_A F|]] < \infty\,, \tag{1.23}$$

for all C if $4\rho < 1$.

Proof. The part in $L_A F$ which comes from (1.22) clearly satisfies (1.23). The part in $L_A F$ which comes from the first sum in (1.18) easily satisfies (1.22). Namely only the derivative of the parallel transport poses any difficulties, but these are overcome by (1.19) and by recognising a non-anticipative Itô integral as in (1.22). It remains to treat the contribution of the counterterms in (1.21). Let us study for instance the contribution of

$$\sum_{n>0,i} A(n)^2 \langle \mathrm{d}F, X(n, e(i))\rangle \int_{[0,1]} \langle S_{X(n,e(i))(s)}, \delta\gamma(s)\rangle$$

$$= \sum_j \int_{[0,1]} \langle S_{Y(j)(s)}, \delta\gamma(s) \rangle , \tag{1.24}$$

where $Y(j)$ is a process of the same type as (1.22). This non-anticipative integral is in particular exponentially integrable. The same holds for the last counterterm.

1.3. Localisation

We can now handle the following theorem, which could justify that the equivariant Euler number under the geometrical action of h should be localised over the twisted loop in g of the fixed point of h.

Theorem 1.6.

$$\mu_g[\mathrm{d}(w_t(\gamma), \gamma) > \delta] < \exp[-C/t] , \tag{1.25}$$

when t is tending to 0.

Proof. d is the uniform distance. Let us cut the time interval in t^{-1} time intervals $[s(i), s(i+1)]$ of the same length. The event $d(w_t(\gamma), \gamma) > \delta$ is included in the union of the events $\{d(w_t(\gamma)(s(i)), \gamma(s(i))) > \delta\} = O_i$ and of the events

$$\{\sup_{[s(i),s(i+1)]} d(w_t(\gamma)(s), w_t(\gamma)(s(i))) > \delta''\} = O'_i .$$

By the stationarity of the process:

$$\exp[-C/t] \geqslant \mu_g\{\sup_{[s(i),s(i+1)]} d(\gamma(s), \gamma(s(i))) > \delta''\} \geqslant \mu_g\{O'_i\} . \tag{1.26}$$

Since the number of O'_i is controlled by t^{-1}, the second term is controlled by $\exp[-C/t]$ when $t \to 0$. Let us estimate $\mu_g\{O_i\}$. By covering M by a set of small balls, O_i can be included in a finite set of $O_{i,j}$ such that:

$$O_{i,j} = \{ | g_j(w_t(\gamma)(s(i))) - g_j(\gamma(s(i))) | > \delta'' \} . \tag{1.27}$$

The g_j are independent of the s_i and δ'' too. Since $g_j(\gamma(s(i))$ belongs to the domain of L_A, quasi-surely, we have:

$$g_j(w_t(\gamma(s(i)))) - g_j(\gamma(s(i))) = M_t + \int_{[0,t]} (L_A g_j)(w_s(\gamma)) \, \mathrm{d}s . \tag{1.28}$$

M_t is a martingale whose derivative of the right bracket is smaller than C because we take a coordinate function. Therefore:

$$\mu_g[|M_t| > C] < \exp[-C/t] . \tag{1.29}$$

Moreover, by the Jensen inequality,

$$\mu_g\left[\exp\left[\int_{[0,t]} L_A g_j\,|\,(w_s(\gamma))\,\mathrm{d}s/t\right]\right]$$

$$<C\mu_g\left[\int_{[0,t]} \exp[\,|L_A g_j|\,(w_s(\gamma))\,]\,\mathrm{d}s/t\right], \tag{1.30}$$

for the stationarity of $w_s(\gamma)$. We deduce from this that

$$\mu_g\left[\int_{[0,t]} |L_A g_j(w_s(\gamma))|\,\mathrm{d}s>C\right]<\exp[-C/t]\,. \tag{1.31}$$

Therefore the result.

2. Regularised Dixon–Harvey–Vafa–Witten Euler number from the loop space of a developable orbifold

2.1. Regularised de Rham operator over the twisted loop space

Let γ be a twisted loop in H_g, and T_γ be its tangent space with the previous Hilbert structure. $T_\gamma=\sum T_\gamma^n$, the sum being taken over the relative integers. Let ΛT_γ be the exterior algebra associated to T_γ with the structure coming from each T_γ^n. The connection Γ passes to ΛT_γ. Let A_g^0 be the set of sections of the shape

$$\sigma=\sum F_I(\gamma(t(0),\gamma(t(0)),...,\gamma(t(r)))X(I)(\gamma)$$

for a finite sum, where F_I is a cylindrical functional and where $X_I=X(n_1)(e_1)\wedge\cdots\wedge X(n_r)(e_r)$. Let us remark that $\Lambda(T_\gamma)$ is canonically isomorphic to $\Lambda(\gamma_0)\wedge\Lambda(\gamma_0,H)$, where $\Lambda(\gamma_0,H)$ is the Fermionic Fock space associated to the L^2 Hilbert structure endowed with the flat Brownian bridge in the tangent space of the starting point. Modulo this isomorphism, we take random sections which have only a finite number of components which are not equal to zero in this bundle over M in order to define A_g^0 and the coordinates are cylindrical functionals. For σ belonging to A_g^0, if $e(i)$ is a local section of the orthonormal basis of $T_{\gamma(0)}M$, we define $d_{r,g}$ by

$$\begin{aligned} d_{r,g}\sigma=\sum_I\Big\{&\sum_{(n,i)} A(n)\langle \mathrm{d}F_I(\gamma(t(0)),...,\gamma(t(r))),X(n,e(i))(\gamma)\rangle\\ &\times X(n,e(i))(\gamma)\wedge X(I)(\gamma)\\ &+\sum_i A(0)F_I(\gamma(t(0),...,\gamma(t(r))X(0,e(i))\wedge\Gamma_{X(0,e(i))(\gamma)}X(I)(\gamma)\Big\}\,. \end{aligned} \tag{2.1}$$

The first sum is taken over the finite number I of components F_I of σ in the distinguished basis $X(I)(\gamma)$ of the exterior algebra ΛT_γ and the second is involved with the derivatives along the distinguished vector fields of the form. Since the connection Γ is a connection which preserves the metric over ΛT_γ, we can write (2.1) more concisely:

$$d_{r,g}\sigma = \sum_i A(n)X(n,e(i)) \wedge \Gamma_{X(n,e(i))}\sigma . \tag{2.2}$$

The operator does not depend on the choice of the local smooth section of the orthonormal basis $e(i)$ we choose. In a particular case, it can be useful to choose a normal system of coordinates in order to determine the operator. We can compute $d^*_{r,g}$ over A^0_g. Namely,

$$\begin{aligned}\mu_g[\langle \mathrm{d}\langle \sigma, \sigma' \rangle, X \rangle] &= \mu_g[\langle \sigma, \sigma' \rangle \operatorname{div} X] \\ &= \mu_g[\langle \Gamma_X\sigma, \sigma' \rangle + \langle \sigma, \Gamma_X \sigma' \rangle] .\end{aligned} \tag{2.3}$$

Therefore

$$\Gamma^*_X\sigma = -\Gamma_X\sigma + \sigma \operatorname{div} X . \tag{2.4}$$

This allows us to show that:

$$d^*_{r,g} = -\sum A(n)\Gamma_{X(n,e(i))} i_{X(n,e(i))}\sigma + \sum i_{X(n,e(i))}\sigma \operatorname{div} X(n, e(i)) . \tag{2.5}$$

Let us recall (see [JL2]) that the sum in (2.1) is infinite but converges because $2\rho<1$ and that in $d^*_{r,g}\sigma$ the sum is finite. $d_{r,g}+d^*_{r,g}$ is a symmetric operator and therefore closable, and $d^*_{r,g}$ is closable too.

Let us show that $(d_{r,g}+d^*_{r,g})^2$ is defined over A^0_g. For this we have to suppose $4\rho<1$. We have:

$$\begin{aligned} d_{r,g}d^*_{r,g}\sigma = \sum_{(n,i)} & A(n)X(n,e(i)) \wedge \Gamma_{X(n,e(i))} \\ & \times \Big\{ -\sum_{(m,j)} A(m)\Gamma_{X(m,e(j))} i_{X(m,e(j))}\sigma \\ & + \sum_{(m,j)} A(m) i_{X(m,e(j))}\sigma \operatorname{div} X(m,e(j)) \Big\} . \end{aligned} \tag{2.6}$$

The sum in { } is in fact finite. We only have to show that if we take $\langle \mathrm{d}F_I, X(n, e(i))\rangle$, we can reach this from the core of cylindrical functionals because the parallel transport appears in such expressions. This comes from the Bismut formula [Bi2, Le4, Le5]

$$\Gamma_X \tau_t = \tau_t \int_{[0,t]} \tau_s^{-1} \mathrm{R}(\mathrm{d}\gamma_s, X_s)\tau_s \tag{2.7}$$

and from the fact that $2p < 1$.

Let us now study the behaviour of $d^*_{r,g} d_{r,g}\sigma$. It equals

$$\sum A(n) \operatorname{div} X(n, e(i)) i_{X(n,e(i))} \left\{ \sum A(m) X(m, e(j)) \wedge \Gamma_{X(m,e(j))}\sigma \right\}$$

$$- \sum A(n) \Gamma_{X(n,e(i))} i_{X(n,e(i))} \left\{ \sum A(m) X(m, e(j)) \wedge \Gamma_{X(m,e(j))}\sigma \right\}. \tag{2.8}$$

The sum is finite, except for the most embarassing term which is

$$\sum A(n)^2 \operatorname{div} X(n, e(i)) \Gamma_{X(n,e(i))}\sigma - \sum A(n)^2 \Gamma_{X(n,e(i))} \Gamma_{X(n,e(i))}\sigma. \tag{2.9}$$

But if we work in a local chart, we can compare the problem of the convergence of this series to the problem of the convergence of L_A and show it is converging in $\mathrm{L}^2(\mu_g)$ as in the first part since $4p < 1$.

The sum in $d^*_{r,g} d^*_{r,g}\sigma$ is finite and does not pose any problem of convergence. The sum in $d_{r,g} d_{r,g}\sigma$ is infinite but its convergence comes from the fact that

$$\mu_g[\,|\langle \mathrm{d}\langle \mathrm{d}F_I, X(n, e(i))\rangle, X(m, e(j))\rangle|^2]$$
$$\leqslant C/(n^2+1)(m^2+1), \tag{2.10}$$

using (2.7).

Remark. Let ω be the form over the twisted loop space which to a vector associates $\langle \omega(\gamma(s)), X_s\rangle$. It is the reciprocal image of the one-form ω in M by the evaluation map which associates to a twist loop its value in time s. It belongs to the domain of $d_{r,g}$ and $d^*_{r,g}$. For this, we expand X_s in the basis given by T^n_γ and we see that this form is the series

$$\sum \langle \omega(\gamma(s)), X(n, e(i))(s)\rangle X(n, e(i)).$$

There is the parallel transport which appears in $\langle \omega(\gamma(s)), X(n, e(i))(s)\rangle$ but can be handled by formula (2.7), which allows us to show that this form belongs to the domain of $d_{r,g}$ and of $d^*_{r,g}$ because $2p < 1$.

The Laplacian $(d_{r,g} + d^*_{r,g})^2 = \Delta_{r,g}$ is densely defined and symmetric, therefore closable.

Let us introduce the geometrical action h: to a twisted loop $\gamma(s)$ it associates the twisted loop $h\gamma(s)$. It is an isometry from H_g into $H_{hgh^{-1}}$ which preserves the splitting of T_γ into $\sum T^n_\gamma$. This comes from the fact that h is an isometry, and if γ is the Brownian bridge between x and gx, $h\gamma$ is the bridge between hx and $hgx = (hgh^{-1})hx$. Moreover the parallel transport between $h\gamma(0)$ and $h\gamma(t)$ is nothing else than $\mathrm{d}h\,\tau_t(\mathrm{d}h)^{-1}$ since h is an isometry. Therefore the isometry between T_γ^n and $T_{h\gamma}^n$ is given by $e \to \mathrm{d}h\,e$. The most difficult part to see this is for a vector field of the type $\tau_t(e - te + t\tau_1^{-1}\,\mathrm{d}g\,e)$. It is transformed in a vector of the type

$$\begin{aligned}
&\tau_t(h\gamma)(\mathrm{d}h\,e - t\,\mathrm{d}h\,e + t\,\mathrm{d}h\,\tau_1^{-1}\,\mathrm{d}g\,e)\\
&\quad= \tau_t(h\gamma)(\mathrm{d}h\,e - t\,\mathrm{d}h\,e + t\,\mathrm{d}h\,\tau_1^{-1}(\mathrm{d}h)^{-1}\mathrm{d}h\,\mathrm{d}g\,(\mathrm{d}h)^{-1}\,\mathrm{d}h\,e)\\
&\quad= \tau_t(h\gamma)(\mathrm{d}h\,e - t\,\mathrm{d}h\,e + t(\mathrm{d}h\,\tau_1\,\mathrm{d}h^{-1})^{-1}\mathrm{d}(hgh^{-1})\,\mathrm{d}h\,e)\,. \qquad (2.11)
\end{aligned}$$

The conclusion follows from $\mathrm{d}h\,\tau_1\,\mathrm{d}h^{-1} = \tau_1(h\gamma)$.

Moreover, h lifts to an application from A_g^0 to $A_{hgh^{-1}}^0$, which is an isometry for the natural L^2 structure over these two spaces. Since h preserves the splitting of T_γ into the sum of T_γ^n, *since the $A(n)$ are independent of the chosen starting point*, and since h preserves the Levi-Civita connection over TM, we deduce the following equalities of operators with their domain:

$$\begin{aligned}
&hd_{r,g} = d_{r,hgh^{-1}}h\,,\\
&hd_{r,g}^* = d_{r,hgh^{-1}}^*h\,,\\
&h(d_{r,g} + d_{r,g}^*) = (d_{r,hgh^{-1}} + d_{r,hgh^{-1}}^*)h\,,\\
&h\Delta_{r,g} = \Delta_{r,hgh^{-1}}h\,. \qquad (2.12)
\end{aligned}$$

The Hilbert space of the loop space of a developable orbifold can be identified with the quotient of the union of the sectors H_g by the geometrical action of G over the union of H_g. Therefore, formally, the Euler–Poincaré characteristic of this orbifold of twisted loop space is given by $(1/|G|)\times \sum \mathrm{Tr}_s(\exp[-t\Delta]h)$, the sum being taken over the elements of the group and the expression Tr_s being the difference of the trace over positive forms and of that over odd forms [HZ]. This quantity is formally equal to $(1/|G|)\sum \mathrm{Ind}_h(d_r + d_r^*)$.

But Δ preserves each fermionic sector, and so only the contribution of the sectors which are kept by the geometrical action of h need to be taken in the equivariant index (it is the diagonal contribution of h). A sector is kept by h if $gh = hg$. So only the sum over commuting pairs (g, h) has to be taken in the Hirzebruch formula.

We can now handle the following conjecture:

Conjecture. If $A(n) > |n|^p$
- $d_{r,g} + d_{r,g}^*$ has a self-adjoint extension.
- If g and h commute, $\chi(M^g \cap M^h) = \mathrm{Ind}_h(d_{r,g} + d_{r,g}^*)$.
- $\mathrm{tr}\exp[-t\Delta_{r,g}]h$ is finite and $\mathrm{Ind}_h(d_{r,g} + d_{r,g}^*) = \mathrm{Tr}_s\exp[-t\Delta_{r,g}]h$.

This conjecture could show that the regularised Euler number of the loop space of a developable compact orbifold is given by that of Dixon–Harvey–Vafa–Witten, Eq. (0.1), given in Section 0.

2.2. A heuristic proof of the conjecture

Over H_g, instead of putting the measure

$$\frac{1}{\int_M p_1(x, gx)\,\mathrm{d}x} p_1(x, gx) P_{1,x,gx}\,\mathrm{d}x = \mu_{1,g}\,,$$

we choose the measure in small time

$$\frac{1}{\int_M p_{\epsilon^2}(x, gx)\,\mathrm{d}x} p_{\epsilon^2}(x, gx) P_{\epsilon^2,x,gx}\,\mathrm{d}x = \mu_{\epsilon,g}\,.$$

When ϵ is small this measure concentrates to the fixed point M^g of g because $p_{\epsilon^2}(x, gx) \leqslant \exp[-C/\epsilon^2]$ when $x \neq gx$ (see [Bi4]). As in [JL2], we divide the metric in T^n_γ, $n \neq 0$, by ϵ^{-2} such that an original orthonormal basis is multiplied by ϵ, although it is kept as a form (see [Bi7] and [Le3]). The contribution of T^0_γ is more complicated to handle, because there are two parts in T^0_γ: the part which is transverse to the fixed point set and the part which is tangent to the fixed point set. Of course this distinction works only if $\gamma(0)$ is close to the fixed point set. If $\gamma(0)$ is close to the fixed point set, we can define the projection $\Pi\gamma(0)$ over the fixed point set and the parallel transport $\tau(\gamma(0), \Pi\gamma(0))$ from $\Pi\gamma(0)$ to $\gamma(0)$. Over M^g, we have the tangent bundle TM^g and its orthogonal bundle $(TM^g)^H$, which are parallel because M^g is totally geodesic. We use the parallel transport $\tau(x, \Pi x)$ in order to get a bundle T_gM and a bundle T_gM^H over a small tubular neighbourhood of the fixed point set M^g. Moreover, T_gM and T_gM^H are orthogonal. If $\gamma(0)$ is in the small neighbourhood of the fixed point set, we can split T^0_γ in $T^0_\gamma(T_gM)$ and $T^0_\gamma(T_gM^H)$. This decomposition is orthogonal. We keep the Hilbert structure in $T^0_\gamma(T_gM)$ and in $T^0_\gamma(T_gM^H)$, we take the Hilbert structure as a $((1-\epsilon^2)f((\gamma(0))/\epsilon^2)+1) = f_\epsilon(\gamma(0))$ multiple of the Hilbert structure from the previous part; moreover $f(\gamma(0))$ is smooth $\geqslant 0$, depends only on the distance between the starting point and the fixed point set M^g, and is equal to 0 outside a small tubular neighbourhood U_1 of the fixed point set and is equal to 1 inside a smaller tubular neighbourhood U_2 of the fixed point set. For the limit theorem we will give later, only the contribution of a small neighbourhood of the fixed point will be significant: an orthonormal basis of $T^0_\gamma(T_gM)$ is the same and an orthonormal basis of $T^0_\gamma(T_gM^H)$ is rescaled by ϵ. We will not write later all the details which come from the fact that this rescaling is only true in fact over a small tubular neighbourhood of M^g, by doing a suitable partition of unity associated to a neighbourhood of the fixed point set invariant under the geometrical action of h.

These definitions being given, we define the operator $d_{\epsilon,r,g}$, the operator $d^*_{\epsilon,r,g}$, the symmetric operator $d_{\epsilon,r,g} + d^*_{\epsilon,r,g}$ and the operator $(d_{\epsilon,r,g} + d^*_{\epsilon,r,g})^2 = \Delta_{\epsilon,r,g}$ as before. Moreover, since we choose $f_\epsilon(x)$ depending only on the distance from x to M^g, and since that distance is invariant under the action of h, because h and g commute, all these operators can be chosen invariant under the action of h: the main difficulty is to show that the splitting into T_gM and T_gM^H is invariant under the action of h. But since h and g commute, h keeps M^g and therefore $\mathrm{d}h$ keeps the decomposition over M^g of TM in TM^g and $(TM^g)^H$. Moreover $\Pi h\gamma(0) = h\Pi\gamma(0)$ always because h and g commute. Moreover, let e_0 be a section of TM^g. $\tau(\gamma(0), \Pi\gamma(0))e_0(\Pi\gamma(0))$ is a section of T_gM. We have:

$$\begin{aligned}
&\mathrm{d}h\, \tau_t\{(1-t)\tau(\gamma(0), \Pi\gamma(0))e_0(\Pi\gamma(0)) \\
&\qquad + t\tau_1^{-1}\, \mathrm{d}g\, \tau(\gamma(0), \Pi(\gamma(0))e_0(\Pi\gamma(0))\} \\
&= \tau_t(h\gamma)\{(1-t)\, \mathrm{d}h\, \tau(\gamma(0), \Pi\gamma(0))e_0(\Pi\gamma(0)) \\
&\qquad + t\, \mathrm{d}h\, \tau_1^{-1}(\mathrm{d}h)^{-1}\mathrm{d}(hgh^{-1})\, \mathrm{d}h\, \tau(\gamma(0), \Pi\gamma(0))(\mathrm{d}h)^{-1}\, \mathrm{d}h\, e_0(\Pi\gamma(0))\} \\
&= \tau_1(h\gamma)\{(1-t)\tau(h\gamma(0), \Pi h\gamma(0))\, \mathrm{d}h\, e_0(\Pi\gamma(0)) \\
&\qquad + t(\tau_1(h\gamma))^{-1}\, \mathrm{d}g\, \tau(h\gamma(0), \Pi h\gamma(0))\, \mathrm{d}h\, e_0(\Pi\gamma(0))\},
\end{aligned} \tag{2.13}$$

and $\mathrm{d}h\, e_0(\Pi\gamma(0))$ is a vector in $\Pi h\gamma(0)$ tangent to M^g. This shows that our splitting is kept near M^g.

We follow the line of [JL2] in order to define the Bismut dilatation. We have our basis of distinguished vector fields $X(n, e(i))$ for a local smooth section $e(i)$ of orthonormal basis. Moreover, over our little neighbourhood of the fixed point set, we choose that local section with respect to the splitting of TM in T_gM and T_gM^H. We deduce from this an orthonormal basis $X(I)$ of the fibre of differential forms. Moreover this choice is invariant under the action of h, because h keeps the splitting. Let us choose the coordinate of $X(I)$. If $d(\gamma(0), M^g)$ is small, we take any finite sum of products of the type

$$f(\Pi\gamma(0)) \prod_{I(n)} (f_i(\gamma(t(i)) - f_i(\Pi\gamma(0))) = F.$$

$I(n)$ is a finite part of cardinal n of $[0, 1]$. Moreover, all the $I(n)$ with the same cardinal are distinct. Moreover, if $I(n) = t(1) < t(2) \cdots < t(n)$, we suppose that the union of all the $I(n)$ for n fixed is dense in the simplex $t(1) < t(2) \cdots < t(n)$ of $[0, 1]^n$. If F is such a functional, $F(h\gamma)$ is still such a functional because $h(\Pi\gamma(0)) = \Pi(h\gamma(0))$, which shows us that the choice of such test functionals is invariant under the action of h, if $d(\gamma(0), M^g)$ is small. If $d(\gamma(0), M^g)$ is large, we take any cylindrical functional. (We do not write completely the details about this, but we stick together the two components by using as test functionals $h(\gamma(0))F(\gamma) + (1 - h(\gamma(0)))G(\gamma)$, where h is a smooth function with compact support in a small neighbourhood invariant under the action of h and equal to 1 in a smaller neighbourhood invariant under the action of h. We perform the Bismut dilatation only over the first component.) Let us suppose that

$$\sum_I f_{0,I}(\Pi\gamma(0))\prod_I (f_{i,I}(\gamma(t_i))-f_{i,I}(\Pi\gamma(0)))=0\,.$$

Since all the I are distinct, we deduce that each $f_{0,I}(\Pi\gamma(0))\Pi(f_{i,I}(\gamma(t_i))-f_{i,I}(\Pi\gamma(0)))$ is equal to 0. We can now define the Bismut dilatation over a functional $F=\sum f_{0,I}\Pi(f_{i,I}(\gamma(t_i))-f_{i,I}(\Pi\gamma(0)))$ by putting:

$$B_\epsilon F=\sum f_{0,I}(\Pi\gamma(0))\prod (f_{i,I}(\gamma(t_i))-f_{i,I}(\Pi\gamma(0))/\epsilon\,. \tag{2.14}$$

If $d(\gamma(0),M^g)$ is large, we do not change the functional. We have that key property, since $\Pi h\gamma(0)=h\Pi\gamma(0)$:

$$B_\epsilon(F(h\gamma))=(B_\epsilon F)(h\gamma)\,. \tag{2.15}$$

Let us show that the space of scalar functionals where the Bismut dilatation is defined is dense. This follows from:

$$\begin{aligned} &f(\Pi\gamma(0))\prod_{I(n)} (f_i(\gamma(t_i))-f_i(\Pi\gamma(0)))\\ &\quad=f(\Pi\gamma(0))\left(\prod_{I(n-1)} (f_i(\gamma(t_i))-f(\Pi\gamma(0)))\right)f_n(\gamma(t_n))\\ &\qquad-f(\Pi\gamma(0))f_n(\Pi\gamma(0))\prod_{I(n-1)} (f_i(\gamma(t(i)))-f_i(\Pi\gamma(0)))\,. \end{aligned} \tag{2.16}$$

By induction over n, we suppose that each functional $f(\Pi\gamma(0),\gamma(t(1)),\ldots,\gamma(t(n-1)))$ is the limit of a sum of finite products with the cardinal of $I(k)$ smaller than $n-1$. If we use this induction hypothesis, it results from (2.16) that we can get any functional of the type $f(\Pi\gamma(0),\ \gamma(t(1)),\ \ldots,\ \gamma(t(n-1)))f_n(\gamma(t(n)))$ in $L^2(\mu_g)$, and therefore all the functionals which are in $L^2(\mu_g)$ by the Stone–Weierstrass theorem.

Let us define the Bismut dilatation for forms: we choose an orthonormal basis $e_i(\Pi\gamma(0))$ of T_gM and an orthonormal basis $e_i(\Pi\gamma(0))$ of T_gM^H. We deduce a basis $X(I)$ of our fibre of differential forms. If we change the orthonormal basis $e_i(\Pi\gamma(0))$, the change of basis $X(I)$ is seen only by terms which depend only on $\Pi\gamma(0)$. If $\sigma=\sum F_IX_I$, let us define

$$B_\epsilon\sigma=\sum (B_\epsilon F_I)X(I)\,. \tag{2.17}$$

This definition is coherent from the remark before. If $d(\gamma(0),M^g)$ is large, there is no operation, and we stick in a smooth way these two operations, but it does not give difficulties, because when ϵ tends to 0, only the contribution of the small tubular neighbourhood of M^g appears.

We have still the basic property:

$$B_\epsilon(\mathrm{d}h\,\sigma)=\mathrm{d}h\,(B_\epsilon\sigma)\,. \tag{2.18}$$

Let us now define the limit model, conformally to [JL2] and [T]. The proba-

bility space is defined as follows:
- Over M^g we take the bundle of bridges in TM which go from c to $\mathrm{d}g\,c$, c being in $(TM^g)^H$. Over M^g, we put the Riemannian measure and over the set of paths which go from c to $\mathrm{d}g\,c$ in time 1, we put the measure $\exp(-\|(I-\mathrm{d}g)c\|^2)\,\mathrm{d}c\otimes P_{1,c,\mathrm{d}g\,c}$, which is the law of the Brownian bridges in T_xM [and not in $(T_xM^g)^H$] which go from c to $\mathrm{d}g\,c$. Let us recall that the Brownian bridges which go from c to $\mathrm{d}g\,c$ have the same law as the process $(1-s)c+s\,\mathrm{d}g\,c+\gamma_{s,\mathrm{flat}}$, which is a flat Brownian bridge starting from 0 and coming back to 0 in T_xM in time 1. The introduction of this model is motivated by [Bi4].

As tangent space of the flat Brownian bridge $\gamma_{s,\mathrm{flat}}$, we take the space H of finite energy elements h of T_xM such that $h(0)=h(1)=0$ with the Hilbert norm $\int_{[0,1]}\|h'(s)\|_x^2\,\mathrm{d}s$. Over the set of c, we take the Hilbert norm $\|c\|^2$. The fact that we use the Hilbert norm $\|c\|^2$ instead of the norm $\|(I-\mathrm{d}g)c\|^2$, which seems more natural, comes from the fact that we use the Hilbert structure $\|e(i)\|^2$ over T_γ^0 for a vector field $X(0,e(i))$. Over an element of that probability limit space, we get as fibre $\Lambda_x\wedge\Lambda_c\wedge\Lambda_{\mathrm{fermionic}}$. The last exterior algebra is the fermionic Fock space associated to the flat Brownian bridge starting from 0 in T_xM.

As limit operator, we choose:

$$\begin{aligned} d_{x,g}+d_{c,g}+d_{\infty,g} &= d_{x,g}+d_{2,l,g}=d_{l,g} \\ &= \sum A(0)e(i)\wedge\Gamma_{e(i)}+\sum A(0)c(i)\wedge\Gamma_{e(i)} \\ &\quad+\sum A(n)(\cos(ns)e(j))\wedge\Gamma_{\cos(ns)e(j)} \\ &\quad+\sum A(n)(\sin(ns)e(j))\wedge\Gamma_{\sin(ns)e(j)}\,. \end{aligned} \tag{2.19}$$

In the first sum, we take derivatives over an orthonormal basis $e(i)$ of T_xM^g (x belongs to M^g). In the second sum, we take derivatives over an orthonormal basis $c(i)$ of $(T_xM^g)^H$ in the limit Gaussian space. In the third sum, we take the classical Araï–Shigekawa complex corresponding to the $A(n)$ and to the Brownian bridge in the full tangent space of M in x starting from 0 and coming back in 0 in time 1. d_x, d_c, d_∞ anticommute as can be seen in normal coordinates. If we work in normal coordinates, we can compute the adjoint of d_l. $d_{l,g}^*$ is given by

$$\begin{aligned} d_x^*+d_c^*+d_\infty^* &= -A(0)\sum i_{e(i)}\Gamma_{e(i)}-A(0)\sum i_{c(i)}\Gamma_{c(i)} \\ &\quad-\sum A(n)\,i_{\cos(ns)e(j)}\Gamma_{\cos(ns)e(j)}-\sum A(n)i_{\sin(ns)e(j)}\Gamma_{\sin(ns)e(j)} \\ &\quad-A(0)\sum i_{c(i)}\langle(I-\mathrm{d}g)c,\,(I-\mathrm{d}g)c(i)\rangle \\ &\quad-\sum\int_{[0,1]}A(n)\langle\cos(ns)e(j),\,\delta\gamma_{\mathrm{flat},s}\rangle \\ &\quad-\sum A(n)\langle\sin(ns)e(j),\,\delta\gamma_{\mathrm{flat},s}\rangle\,. \end{aligned} \tag{2.20}$$

It is the same type of formula as in [JL2], but the normal flat Brownian bridge is more complicated here, because we choose c random too. If we put c and γ_{flat} together, we have an abstract Wiener space, and $d_{c,g}+d_{\infty,g}$ can be understood in the formalism of Araï [Ar1, Ar2, ArM]. Namely, we can choose the $c(i)$ such that it is an orthonormal basis for $(TM^g)^H$ for the norm $\|c\|^2$. Let us recall namely that over M^g, if we write $\mathrm{d}g$ as a collection of matrices of rotation of angle θ, we get a collection of orthogonal subbundles which are parallel over each component of M^g. Modulo this $d_c^* + d_\infty^*$ appears as an Araï operator with an auxiliary operator in c for the Gaussian space spanned by c and the flat Brownian bridge γ. As Fermionic Fock space, we choose $\Lambda_c \wedge \Lambda_\gamma$ with the norm $\|c\|^2$ and as Bosonic Fock space, the space L^2 associated to the limit Gaussian probability measure $\exp(-\|(I-\mathrm{d}g)c\|^2)\otimes \mathrm{d}P_{1,x}$. The auxiliary operator in c is the operator which allows us to pass from both Hilbert structures in c.

Moreover

$$d_{\infty,g}d^*_{\infty,g}+d^*_{\infty,g}d_{\infty,g}=\Delta_{\infty,g}=N_{\mathrm{B}}(A^2)+N_{\mathrm{F}}(A^2)\ ,$$

$$d_{c,g}d^*_{c,g}+d^*_{c,g}d_{c,g}=\Delta_{c,g}=N_{\mathrm{B}}(c^2)+N_{\mathrm{F}}(c^2)\ .$$

The number operator for bosons $N_{\mathrm{B}}(A^2)$ is associated to the operator which sends $\sin(ns)$ to $A(n)^2\sin(ns)$ and $\cos(ns)$ to $A^2(n)\cos(ns)$ as well as the fermion number operator $N_{\mathrm{F}}(A^2)$. The bosonic number operators $N_{\mathrm{B}}(c^2)$ and the fermionic number operators $N_{\mathrm{F}}(c^2)$ are related to the change of Hilbert structure in $(TM^g)^H$.

Therefore, $d_{l,g}+d^*_{l,g}$ has a self-adjoint extension. Namely $\Delta_{c,g}+\Delta_{\infty,g}$ can be diagonalised, because it is a sum of bosonic number operators and of fermionic operators (see [Ar2]). Since the $A(n)$ do not depend on x and since the diagonalisation of $\mathrm{d}g$ is parallel over M^g, we deduce that the set of eigenvectors associated to different eigenvalues of $\Delta_{c,g}$ and of $\Delta_{c,\infty}$ constitute a countable set of finite dimensional bundles over M^g, which are preserved by $d_{x,g}+d^*_{x,g}$, $d_{c,g}+d_{c,g}$ and $d_{\infty,g}+d^*_{\infty,g}$, this because these operators are anticommuting (cf. [T] and [JL2]). $d_{x,g}+d^*_{x,g}$ appears exactly over each bundle as the de Rham operators tensorised by this bundle. We know that the spectrum of $d_{x,g}+d^*_{x,g}$ is discrete over each of these finite dimensional bundles, as well as $d_{c,g}+d^*_{c,g}$ and $d^*_{\infty,g}+d_{\infty,g}$. Moreover the action of $d_{c,g}+d^*_{c,g}$ and of $d_{\infty,g}+d^*_{\infty,g}$ over each of these bundles is the square root of the action modulo the sign of $\Delta_{c,g}$ and of $\Delta_{\infty,g}$ over each of these bundles. This allows us by restricting these bundles to diagonalise $d_{l,g}+d^*_{l,g}$ and to show it has a self-adjoint extension.

We can look at the action of h over the limit model. h keeps M^g because g and h commute. Moreover, $\mathrm{d}h$ lifts over M^g to a natural action over TM, which preserves TM_g and $(TM_g)^H$. Let us remark that:

$$\|\mathrm{d}h(I-\mathrm{d}g)c\|^2=\|(I-\mathrm{d}g)(\mathrm{d}h\,c)\|^2\ , \tag{2.21}$$

such that the action of $\mathrm{d}h$ preserves the auxiliary operator which appears in $\Delta_{c,g}$

and $\Delta_{\infty,g}$ (since the action of $\mathrm{d}h$ preserves the metric of the tangent space of γ_{flat}). This shows us that $\mathrm{d}h$ commutes with all the limit operators given before.

Theorem 2.1. *If $A(n) > |n|^p$, then $gh = hg$,*

$$\operatorname{Tr}\exp(-t\Delta_{l,g}h) < \infty\,, \tag{2.22}$$

$$\operatorname{Ind}(d_{l,g}+d^*_{l,g})h = \chi(M^g \cap M^h)\,. \tag{2.23}$$

Proof. The proof of the existence of the trace follows directly the line of [JL2], because h keeps the Wick product and the fermionic Fock space. Let Ξ_K be such a subbundle for $\Delta_{c,g}$ and $\Delta_{\infty,g}$ endowed with a given combination of Wick products in $\sin(ns)$, $\cos(ns)$, and of exterior algebra in $\cos(ns)$ and $\sin(ns)$. K denotes the combination of $\sin(ns)$, $\cos(ns)$ which appear in Ξ_K: it is possible that more than one of each $\sin(ns)$ appear there. $|K|$ is the cardinal of K. The dimension of such a subbundle is bounded by $C^{|K|+1}$, and the action of $\exp(-t\Delta_{2,l,g})$ over each subbundle is diagonal and bounded by $C\exp(-t\sum_K A(n)^2)$. The action of Δ_x over $\Lambda_x \wedge \Xi_K$ is given by the Lichnerowicz formula $\Delta_K = -\frac{1}{2}\Delta^H M^g + C_1 + C_K$. C_1 is the action of the Lichnerowicz formula for the non-tensorised de Rham operator, and C_K comes from the action of the Lichnerowicz formula over the auxiliary bundle, which appears as a combination of at most $|K|$ products of three types together: exterior products, symmetric tensor products and tensor products. We have a probabilistic representation of the trace of the heat semi-group associated to Δ_K, since over each product we take the connection product. Let $\tau_{s,K}$ be the parallel transport over Ξ_K, which preserves the product, and let

$$\mathrm{d}U_{s,K} = -\tfrac{1}{2}U_{s,K}\tau^{-1}_{s,K}(C_K + C_1)\,. \tag{2.24}$$

We get the following representation of [Bi3, IW, Le2, Le3] and more precisely [Bi4] of the trace that the heat semi-group:

$$\operatorname{Tr}\exp(-t\Delta_K)h = \int_{M^g} p_t(x, hx)E_{t,x,hx}(\operatorname{tr}(U_{1,K}\tau^{-1}_{1,K}\,\mathrm{d}h))\,\mathrm{d}x\,, \tag{2.25}$$

where $p_t(x, y)$ is the heat kernel associated to the Brownian motion over M^g and $E_{t,x,h(x)}$ the expectation for the Brownian bridge which goes from $h(x)$ to x in time t. In particular, we have a bound of the trace under the expectation in C,

$$\begin{aligned} C^{(1+t)|K|}\prod_K \exp(-t|A(n)|^2) &< \sum C^{|K|(1+t)}\prod_K \exp(-tn^{2p}) \\ &= C\prod_K C^{(1+t)}\exp(-tn^{2p}) < \infty\,. \end{aligned}$$

This shows that the first part of the theorem is true.

Let us show now that the second part of the theorem is true. The operators $d_{\infty,g}$,

$d_{c,g}$, $d_{x,g}$ anticommute or commute with h. If a section belongs to the kernel of $d_{l,g}+d^*_{l,g}$, it is therefore, by using Araï's computation [Ar1, Ar2], a form in x which does not depend on γ_{flat} and c, almost surely. This shows us that

$$\operatorname{Ind}_h(d_{l,g}+d^*_{l,g})=\chi_h(M^g) . \tag{2.26}$$

We apply the classical Lefschetz theorem and we find

$$\chi_h(M^g)=\chi(M^g\cap M^h) , \tag{2.27}$$

since h is an isometry of M^g because g and h commute.

Let us now motivate the introduction of these operators by the following limit theorem, which is analogous to the limit theorem of [JL2]. But before this, we need to understand what we mean by a limit in law, because our situation is a little bit more complicated than the situation encountered in [JL2]. Let us recall that the fibre is isomorphic to $\Lambda(T_xM)\wedge\Lambda_x(H)$. But if x is close to the fixed point set, $\Lambda(T_x)\wedge\Lambda_x(H)$ is isomorphic by means of the parallel transport between x and Πx to $\Lambda(T_{\Pi x})\wedge\Lambda_{\Pi x}(H)$. We identify the fibre close to M^g with $\Lambda(T_{\Pi x})\wedge\Lambda_x(H)$ and far from M^g with the original fibre. We put as Hilbert space structure the space of L^2 sections over $\Lambda(T_{\Pi x})\wedge\Lambda_{\Pi x}(H)$ and the space of L^2 sections over $\Lambda(T_x)\wedge\Lambda_x(H)$ far from our neighbourhood. An L^2 section of forms over the twisted loop space appears therefore as an L^2 random variable from the twisted loop space into this fixed Hilbert space. It makes sense in particular to speak of the limit in law of such a random variable into this fixed Hilbert space, which justifies our conjecture.

Theorem 2.2. *For any fixed element of A^0_g, we have in law if $4p<1$:*

$$B_\epsilon\sigma\to\sigma_l , \tag{2.28}$$

$$B_\epsilon\, \mathrm{d}h\,\sigma\to\mathrm{d}h\,\sigma_l , \tag{2.29}$$

$$(d_{\epsilon,r,g}+d^*_{\epsilon,r,g})\,\mathrm{d}h\,B_\epsilon\sigma\to(d_{l,g}+d^*_{l,g})\,\mathrm{d}h\,\sigma_l , \tag{2.30}$$

$$\Delta_{\epsilon,r,g}\,\mathrm{d}h\,B_\epsilon\sigma\to\Delta_{l,g}\,\mathrm{d}h\,\sigma_l . \tag{2.31}$$

Proof. Let us begin to show first that $B_\epsilon\sigma\to\sigma_l$ in law for any element of A^0_g. This arises from the Bismut computation of [Bi4]: any finite combination of $(f(\gamma(t))-f(\Pi\gamma(0)))/\epsilon$ tends in law to $\langle \mathrm{d}f(\gamma(0)), \gamma_{\text{flat}}(t)+(1-t)c+t\,\mathrm{d}g\,c\rangle$ for the given limit probability Gaussian measure. The second affirmation comes from the fact that $\mathrm{d}h\,\sigma$ belongs to A^0_g and that $B_\epsilon\,\mathrm{d}h\,\sigma=\mathrm{d}h\,B_\epsilon\sigma$. Let us remark that we do not need the full Bismut procedure in order to see that, because we take only a finite number of terms in $(f(\gamma(t))-f(\Pi\gamma(0)))/\epsilon$. Computations similar to [Le2] can be used.

Let us show now that $(d_{\epsilon,r,g}+d^*_{\epsilon,r,g})B_\epsilon\sigma$ tends in law to $(d_{l,g}+d^*_{l,g})\sigma_l$. (We can remove the term in h, this from (2.30).) We work in normal coordinates in $\Pi\gamma(0)$. Let us begin by the divergence part in $d^*_{\epsilon,r,g}$. If we take an element of T^n_γ, $n>0$, of the distinguished basis, it is multiplied by ϵ, this from the rescaling of the metric. We get:

$$\begin{aligned}\operatorname{div} \epsilon X(n, e(i)) = \epsilon/\epsilon^2 \int_{[0,1]} \langle \tau_s \cos(ns) e(i), \delta\gamma(s)\rangle \\ + \tfrac{1}{2}\epsilon^2/\epsilon^2 \int_{[0,1]} \langle S_{\epsilon X(n,e(i))(s)}, \delta\gamma(s)\rangle + \text{counterterms}\,.\end{aligned} \tag{2.32}$$

The counterterms disappear when ϵ tends to zero, and in law we obtain at the end $\int_{[0,1]}\langle \cos(ns)e(i), \delta\gamma_{\text{flat}}(s)\rangle$, which is exactly the divergent term which appears in $d^*_{\infty,g}$. In the limit contribution, there is no term in $\int_{[0,1]}\langle\cos(ns)e(i),$ $(I-\mathrm{d}g)c\,\mathrm{d}s\rangle = 0$. The case $n<0$ is similar. For the moment, we do not see a difference with the computation of [JL2].

The difference appears when we want to treat the contribution in the divergent part of T^0_γ, because in this case there are two distinct behaviours.

Let us consider first the case of $X_0(\tau(\gamma(0)), \Pi\gamma(0))e(i)(\Pi\gamma(0))$, where $e(i)$ is orthogonal to TM^g. In this case, the metric is rescaled, and near M^g we have to multiply our vector by ϵ. We get:

$$\begin{aligned}\epsilon \operatorname{div} X(0, e(i)) = \epsilon \operatorname{div} \tau(\gamma(0), \Pi\gamma(0))e(i)(\Pi\gamma(0)) \\ + \epsilon/\epsilon^2 \int_{[0,1]} \langle \tau_s(-\tau(\gamma(0), \Pi\gamma(0))e(i)(\Pi\gamma(0)) \\ + \tau_1^{-1}\,\mathrm{d}g\,\tau(\gamma(0), \Pi\gamma(0))e(i)(\Pi\gamma(0))), \delta\gamma(s)\rangle \\ + \epsilon^2/\epsilon^2 \int_{[0,1]} \langle S_{\epsilon X(0,e(i))}, \delta\gamma(\sigma)\rangle + \text{counterterms}\,.\end{aligned} \tag{2.33}$$

In this case, the limit in law of a finite family of divergences of this kind is $\langle(-1+\mathrm{d}g)e(i)(\gamma(0)), (-1+\mathrm{d}g)c\rangle$, which gives the divergence part of the term in $d^*_{c,g}$ in the limit model.

If we work now over TM^g, we have the same type of behaviour, but this goes in law to $\operatorname{div} e(i)$, because τ_1 has a behaviour in $I+\epsilon^2$ and because $e(i)(\Pi\gamma(0))$ belongs to the kernel of $-1+\mathrm{d}g$. This shows us in local coordinates that the divergence part of $d^*_{\epsilon,r,g}B_\epsilon\sigma$ converges to the divergence part of $d^*_{l,g}\sigma_l$, and this without the Bismut procedure, because in this case we have only a finite expression. (In fact, it is not so simple, because in the limit theorem in law, we take test functionals which are only continuous, and it is not easy to regularise continuous test functions in the non-compact case. In order to be rigorous, we cannot avoid to use the Bismut procedure. See later for this.)

Let us now study the behaviour of $d_{\epsilon,r}B_\epsilon\sigma$. The difficulty is now that we have infinite expressions. If $n>0$, we have to study the behaviour of $\langle d(f(\gamma(t))-f(\Pi\gamma(0)))/\epsilon,\ \epsilon X(n,\ e(i))\rangle$, which is equal to $\langle \mathrm{d}f(\gamma(t)),\ X(n,\ e(i))(t)\rangle$ because $X(n,\ e(i))(0)=0$. This tends to $\langle \mathrm{d}f(\gamma(0)),\ \int_{[0,t]}\cos(ns)\,\mathrm{d}s e(i)\rangle$, which is exactly the derivative of $\langle \mathrm{d}f(\gamma(0)), \int_{[0,t]}\delta\gamma_{\mathrm{flat}}(s)\rangle$ in the direction $\cos ns\, e(i)$ of the Cameron–Martin space H of the Brownian bridge. The case $n<0$ is similar. If we take a derivative in the direction of $(TM^g)^H$, $\Pi\gamma(0)$ does not change asymptotically in ϵ under the action of such a vector field. The vector field is rescaled by ϵ itself, because we rescale the metric in this direction. So we find that in law $\langle \mathrm{d}(f(\gamma(t)))-f(\Pi\gamma(0)),\ X(0,\ e(i))\rangle$ tends to $\langle \mathrm{d}f(\gamma(0)),\ t(-1+\mathrm{d}g)e(i)\rangle$, which is exactly the derivative of $\langle \mathrm{d}f(\gamma(0)),\ t(-1+\mathrm{d}g)c\rangle$ in the direction $e(i)$. Let us now study the behaviour in law of the derivatives in the direction of TM^g. We get, if $e(i)$ belongs to TM^g,

$$\begin{aligned}&\langle \mathrm{d}(f(\gamma(t))-f(\Pi\gamma(0)))/\epsilon, X(0,e(i))\rangle\\ &\quad=\langle \mathrm{d}(f(\gamma(t))-f(\gamma(0))), X(0,e(i))\rangle/\epsilon\\ &\quad+\langle \mathrm{d}(f(\gamma(0))-f(\Pi\gamma(0)), X(0,e(i))\rangle/\epsilon .\end{aligned}\tag{2.34}$$

Since we work in normal coordinates, and since $\tau_1(\gamma(0),\Pi\gamma(0))\approx I+\epsilon^2$ when ϵ tends to 0, the derivative of the second term disappears almost completely when ϵ goes to 0, because $d(\gamma(0),\Pi\gamma(0))\approx\epsilon$ when ϵ goes to 0: it remains in the limit $\langle \Gamma_{e(i)}\,\mathrm{d}f(\gamma(0)),c\rangle$. Let us treat the first term. But it is in the Stratonovitch sense:

$$\begin{aligned}&\left\langle \mathrm{d}\int_{[0,t]}\langle \mathrm{d}f(\gamma(s)),\mathrm{d}\gamma(s)\rangle, X(0,e(i))\right\rangle\Big/\epsilon\\ &\quad=\int_{[0,t]}\langle \Gamma_{X(0,e(i))(s)}\,\mathrm{d}f(\gamma(s)),\mathrm{d}\gamma(s)\rangle/\epsilon\\ &\quad+\int_{[0,t]}\langle \mathrm{d}f(\gamma(s)),\tau_s(-\tau(\gamma(0),\Pi\gamma(0))e(i)(\Pi\gamma(0))\\ &\quad+\tau_1^{-1}\,\mathrm{d}g\,\tau(\gamma(0),\Pi(\gamma(0))e(i)(\Pi\gamma(0)))\rangle/\epsilon .\end{aligned}\tag{2.35}$$

The second term tends to 0 because $\tau_1^{-1}\,\mathrm{d}g=I+\epsilon^2$ in law and at the end we see that:

$$\begin{aligned}&\langle \mathrm{d}(f(\gamma(t))-f(\Pi\gamma(0)), X(0,e(i))\rangle/\epsilon\\ &\quad\to\int_{[0,t]}\langle \Gamma_{e(i)}\,\mathrm{d}f(\Pi\gamma(0)),\delta\gamma_{\mathrm{flat},s}+(-1+\mathrm{d}g)c\,\mathrm{d}s\rangle+\langle \Gamma_{e(i)}\,\mathrm{d}f(\Pi\gamma(0)),c\rangle.\end{aligned}\tag{2.36}$$

Let us remark that the simplifications which appear because we use local normal coordinates over $\Pi\gamma(0)$ (for instance $\tau_t \approx 1+\epsilon^2$) do in general not appear in each part of the operator but only globally. For instance, the derivatives of the distinguished vector fields cancel when ϵ tends to 0 because we use that normal coordinate system in $\Pi\gamma(0)$; without this the computation should be more complicated. The difficulty we have to overcome is that we get in fact an infinite sum. In order to solve that, we will use the Bismut procedure as given in [Bi2, Bi4] and not in [Le1], because in this case some smoothness assumption is necessary about the auxiliary functional of the Brownian bridge which is considered. Let us consider the collection of

$$\left\langle \mathrm{d}f_i(\gamma(t)), \tau_t \int_{[0,t]} \cos(ns)\,\mathrm{d}s\, e(j) \right\rangle, \left\langle \mathrm{d}f_i(\gamma(t)), \tau_t \int_{[0,t]} \sin(ns)\,\mathrm{d}s\, e(j) \right\rangle .$$

It is a random element Φ_ϵ of $\mathrm{L}^2(\mathbb{N})$. We have to show that for all bounded continuous functionals F from $\mathrm{L}^2(\mathbb{N})$ into $\mathbb{R}$, the expectation of $F(\Phi_\epsilon)$ tends to the expectation of $F(\Phi)$. We use the Bismut fact that:

$$\begin{aligned} &\mu_{\epsilon,g}(F(\Phi_\epsilon(\gamma))) \\ &\quad = \mu(F\Phi(x, \epsilon\gamma_{\text{flat}} + \epsilon((1-s)c + \mathrm{d}g\, sc) + \epsilon\nu^2(\epsilon\gamma, c, x))) + \mathrm{o}(\epsilon^2) . \end{aligned} \tag{2.37}$$

In order to get ν^2, we look at the equation

$$\mathrm{d}u_s = \sum X_i(u_s)(\epsilon \mathrm{d}\gamma_{s,\text{flat}} + \epsilon(1-\mathrm{d}g)c\,\mathrm{d}s + \epsilon\nu^2\,\mathrm{d}s) , \tag{2.38}$$

where the X_i are the canonical vector fields over the frame bundle over M. We suppose that u_s starts from $\Pi u(0) = \gamma(0)+\epsilon c$, $\gamma(0)$ belonging to M^g and c being in TM^g, this expression being written in a tubular neighbourhood. We choose ν^2 such that $\gamma(1) = \Pi u_1$ is equal to $\gamma(0)+\epsilon\,\mathrm{d}g\,c$. We start from $\gamma(0)+\epsilon c$, because the heat kernel $p_{\epsilon^2}(x, gx)$ does not tend to zero when x has a behaviour in $\gamma(0)+\epsilon c$ (see [Bi4] for more details). The key fact is the following: generally for a given ϵ, we cannot find a ν^2 such that $\gamma(1) = \gamma(0)+\epsilon\,\mathrm{d}g\,c$, but it is asymptotically true, and this uniquely (see [Le1] for a non-geometrical approach). This explains the error term in (2.37), which is explained as well by the contribution of the Jacobian which appears in the implicit function theorem which tends to 1 when ϵ goes to 0. The last difficulty that remains to explain is that there is a τ_t which is not a continuous functional of γ. But this is overcome because τ_t appears in (2.38). (It works too if we take a finite number of stochastic integrals with different integrands.)

This type of argument works too (but in a simpler way, because in this case there is a finite sum) for the non-divergent part of $d^*_{\epsilon,r,g}B_\epsilon\sigma$. In conclusion, we have shown that in law $(d_{\epsilon,r,g}+d^*_{\epsilon,r,g})B_\epsilon\sigma$ tends to $(d_{l,g}+d^*_{l,g})\sigma_l$.

Let us now show that $(d_{\epsilon,r,g}+d^*_{\epsilon,r,g})^2 B_\epsilon\sigma = \Delta_{\epsilon,r,g}\sigma$ converges in law to $\Delta_{l,g}\sigma_l$.

(a) Let us begin by the simplest contribution, that means $d^*_{\epsilon,r,g}d^*_{\epsilon,r,g}B_\epsilon\sigma$. It is a finite sum which appears. Moreover, since two interior products anticommute and since the derivatives in the limit probability space over $\gamma(0)$ in M^g commute, the limit in law of this expression is nothing else than $d^*_{x,g}d^*_{x,g}\sigma_I$.

(b) Let us look at the contribution of $d_{\epsilon,r,g}d_{\epsilon,r,g}B_\epsilon\sigma$. There is a doubly infinite sum in this expression. Since we look in normal coordinates, we see that the apparently most difficult part to handle in this expression is

$$\sum_{n\neq 0, m\neq 0, i,j} A(m)A(n)\langle \mathrm{d}\langle \mathrm{d}B_\epsilon F_I, \epsilon X(n, e(i))\rangle, \epsilon X(m, e(j))\rangle$$
$$\times X(m, e(j)) \wedge X(n, e(i)) \wedge X(I) .$$

The contribution with $n=m$, $e(i)=e(j)$ cancels. We have only to consider the family of

$$A(m)A(n)\epsilon^2(\langle \mathrm{d}\langle \mathrm{d}B_\epsilon F_I, X(n, e(i))\rangle, X(m, e(j))\rangle$$
$$-\langle \mathrm{d}\langle \mathrm{d}B_\epsilon F_I, X(m, e(j))\rangle, X(n, e(i))\rangle) ,$$

which belongs to $\mathrm{L}^2(\mathbb{N})$. The only difficulty is when we derive twice the same $(f(\gamma(t(i)))-f(\Pi\gamma(0)))/\epsilon$; in the other case, there is an automatic cancellation. We use again the Bismut procedure, but we have to use the formula (2.7). A infinite number of stochastic integrals with different integrands appear. We overcome this difficulty by writing $X_s=\tau_s H_s$ and by integrating by parts in (2.7). The boring terms are of the type $\tau_t\int_{[0,t]} K_s H'_s\,\mathrm{d}s$ where a fixed K_s (independent of H_s) appears. K_s is a Stratonovitch integral in the curvature tensor and τ_s: we can apply the Bismut procedure to K_s, which allows to conclude. Let us remark that the fact that the second derivative of $f(\gamma(t_i))-f(\Pi\gamma(0))/\epsilon$ along $\epsilon X(n, e(i))$ and $\epsilon X(m, e(j))$ cancels at the limit, describes the fact that the derivative of $\langle \mathrm{d}f(\gamma(0)), \gamma_{\mathrm{flat},t}+(1-t)c+t\,\mathrm{d}g\,c\rangle$ is deterministic at the limit in the direction of the tangent space of the Brownian bridge.

In this case, the computation was easier because we divide each $f(\gamma(t(i)))-f(\Pi\gamma(0))$ by ϵ and we multiply each $X(n, e(i))$ by ϵ. This simplification does not appear when we have to multiply only one $X(n, e(i))$ by ϵ. We look at the convergence in law of the series in $\mathrm{L}^2(\mathbb{N})$,

$$A(n)\epsilon\{\langle \mathrm{d}\langle \mathrm{d}B_\epsilon F_I, X(0, e(i))\rangle, X(n, e(j))\rangle$$
$$-\langle \mathrm{d}\langle \mathrm{d}B_\epsilon F_I, X(n, e(j))\rangle, X(0, e(i))\rangle\} ,$$

where $e(i)$ belongs to TM^g, $n\neq 0$, or $e(j)$ belonging to $(TM^g)^H$, $n=0$. Only the contribution of the second derivative of the same $(f(\gamma(t(i)))-f(\Pi\gamma(0)))/\epsilon$ plays a role. We have for $n\neq 0$,

$$\langle \mathrm{d}\langle \mathrm{d}(f(\gamma(t(i)))-f(\gamma(0))), X(0, e(i))\rangle, X(n, e(j))\rangle A(n)$$
$$=\langle \mathrm{d}\langle \mathrm{d}f(\gamma(t)), X(0, e(i))(t)\rangle, X(n, e(j))\rangle A(n) , \tag{2.39}$$

$$\langle \mathrm{d}\langle \mathrm{d}(f(\gamma(t(i))) - f(\Pi\gamma(0))), X(n, e(j))\rangle, X(0, e(i))\rangle A(n)$$
$$= \langle \langle \mathrm{d}f(\gamma(t(i))), X(n, e(j))(t)\rangle, X(0, e(i))\rangle A(n)\,. \tag{2.40}$$

We have to take the derivative $\Gamma_{X(0,e(i))}X(n, e(j))(t)$ and $\Gamma_{X(n,e(j))}X(0, e(i))(t)$. These two sequences tend separately in law to 0, because we work in the normal coordinate system. Let us repeat that this simplification appears separately because we use normal coordinates over each contributing term of the operator and appears globally over the operator, which is intrinsically defined. It remains only to study the contribution of the sequence:

$$A(n)(\langle \Gamma^2 f(\gamma(t)), X(0, e(i))(t), X(n, e(j))(t)\rangle$$
$$- \langle \Gamma^2 f(\gamma(t)), X(n, e(j))(t), X(0, e(i))(t)\rangle)\,, \tag{2.41}$$

which in $\mathrm{L}^2(\mathbb{N})$ tends in law to 0 because we work in normal coordinates. We have shown that $d_{\epsilon,r,g}d_{\epsilon,r,g}B_\epsilon\sigma$ tends in law to $d_{x,g}d_{x,g}\sigma_l$.

(c) We consider the case of $d_{\epsilon,r,g}d^*_{\epsilon,r,g}B_\epsilon\sigma$. Since $d^*_{\epsilon,r,g}B_\epsilon\sigma$ is a finite sum, that term can be treated as the contribution of $d_{\epsilon,r}B_\epsilon\sigma$. The first difference is that we have to take derivatives of the parallel transport, and therefore to use (2.7). We also have to take the derivative of the divergence part. The only difficulty in this case is to take the derivative of $(1/\epsilon)\int_{[0,1]}\langle \tau_s H'(n)(s), \delta\gamma(s)\rangle$. It is in fact also a Stratonovitch integral. We have then, if $m \neq 0$:

$$\left\langle \frac{1}{\epsilon} \int_{[0,1]} \langle \tau_s H'(n)(s), \mathrm{d}\gamma(s)\rangle, \epsilon X(m, e(i))\right\rangle$$
$$= \int_{[0,1]} \langle \tau_s H'(n)(s), \tau_s H'(m)(s)\rangle\, \mathrm{d}s$$
$$+ \int_{[0,1]} \left\langle \tau_s \int_{[0,s]} \tau_u^{-1} R(\mathrm{d}\gamma_u, X(n, e(i))(u)\tau_u H'(n)(s), \mathrm{d}\gamma(s)\right\rangle, \tag{2.42}$$

which tends in law to $\int_{[0,1]}\langle H'(n)(s), H'(m)(s)\rangle\, \mathrm{d}s$, therefore the derivative of the divergence $\int_{[0,1]}\langle H'(n)(s), \delta\gamma_{\text{flat}}(s)\rangle$ in the other direction $H(m)(s)$. The fact that the second term tends in law to zero does not come from the fact that we use normal coordinates, but we have to use this for the derivatives of the other parts of the divergence.

(d) The most complicated term to treat is $d^*_{\epsilon,r,g}d_{\epsilon,r,g}B_\epsilon\sigma$ because the sum in $d_{\epsilon,r,g}B_\epsilon\sigma$ is infinite, among which is the term:

$$\sum_{n\neq 0,i} A(n)^2\epsilon^2\langle \mathrm{d}\langle \mathrm{d}B_\epsilon F_I, X(n, e(i))\rangle, X(n, e(i))\rangle$$
$$- \epsilon^2 \sum A(n)^2 \langle \mathrm{d}B_\epsilon F_I, X(n, e(i))\rangle \operatorname{div} X(n, e(i))\,. \tag{2.43}$$

The first term does not pose any problem, because each term of the series is in L^2 bounded by $CA(n)^2/n^2$ and since $2p<1$. For the second, the most complicated term is

$$D(\epsilon)=\frac{1}{\epsilon}\sum_{n\neq 0,e(i)} A(n)^2\langle \mathrm{d}f(\gamma(t)),\tau_t H(n)(t)\,e(i)\rangle \times \int_{[0,1]} \langle \tau_s H'(n)(s)e(i),\delta\gamma(s)\rangle\,. \tag{2.44}$$

The deterministic series $A(n)^2H_n(t)$ is in $L^2(\mathbb{N})$ because $4p<1$. Let $\phi_t(s)=\sum A(n)^2H(n)(t)H'(n)(s)$. It is an element deterministic of $L^2[0,1]$, which does not depend on ϵ. We recognise in (2.45)

$$D(\epsilon)=\frac{1}{\epsilon}\sum\left\langle \mathrm{d}f(\gamma(t)),\tau_t\int_{[0,1]}\langle\tau_s\phi_t'(s)e(i),\delta\gamma(s)\rangle\right\rangle. \tag{2.45}$$

Since $\phi_t'(s)$ is deterministic, this converges in law to $\sum\langle \mathrm{d}f(\gamma(0)), \int_{[0,1]}\langle\phi_t'(s)e(i),\delta\gamma(s)\rangle\rangle$. Since $\phi_t'(s)$ is deterministic, this converges in law to $\sum\langle \mathrm{d}f(\gamma(0)),\int_{[0,1]}\langle\phi_t'(s),\delta\gamma_{\mathrm{flat}}(s)\rangle$, which is the divergent part of the operator associated to the auxiliary operator which to $H'(n)$ associates $A(n)^2H'(n)$ over the flat Brownian bridge.

Remark. We separate in order to give a nice exposure of the convergence in law of different parts of the considered expression, although it is not completely correct. But the convergence in law for the whole expression together is ensured.

Remark. Theorem 2.2 justifies the name of limit model, although we omit to speak about the difficulties of this limit procedure: it is perhaps possible to define another set of functionals such that the Bismut dilatation gives another limit operator.

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